of Chapter IV, Section 48,

urves in our fixed cube, we
the integrand f. However,
m now, but with the notion
s therefore f that now must

(C), considered for varying
on $g(f)$. Now the quantity
C, has rather simple prop-

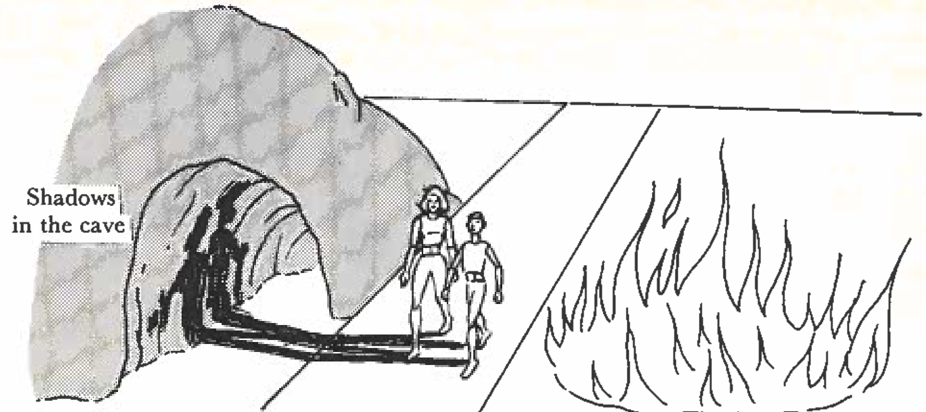
C on a segment $t_1 \leq t \leq t_2$.
are constants and L_1, L_2 are

agrangians L_1, L_2. Since, by
to L, L_1, L_2 are related by
i.e., that g is linear. Further,
it when f lies in the unit ball
annot exceed the length of C.

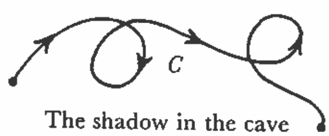
$\mathscr{C}_0^*(A)$. The value $g(f)$ for a
nce with Chapter IV, Section

s function of L, i.e., of f. This
the element g of $\mathscr{C}_0^*(A)$. Ad-
mely those for each of which
$_2$ such that, for each $f \in \mathscr{C}_0(A)$,

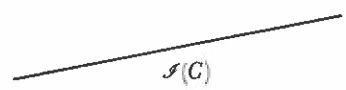
e C with the corresponding opera-
corresponding element $g \in \mathscr{C}_0^*(A)$,
lapse from precision at all). We
have stumbled onto a much better
erstanding of what a curve is.
l, and best described, in terms of
of shadows, cast onto the walls of
along a road behind us.
ctic method: we must learn to re-
ve. This has become the modern
n mathematics. For us, therefore,
makes it a shadow in our cave, nor
This is also the logical sequel to the
omy, by no longer representing the
cepts which are simple in this sense.



The classical concept of a curve C corresponds very closely to the curves that we see and draw; and such curves can twist and turn and zigzag back and forth; moreover they can have highly complicated self-intersections. This is only the shadow. If we choose to substitute for it the notion of curvilinear integral $\mathscr{I}(C)$, it will become an element g of our dual space $\mathscr{C}_0^*(A)$. This element, of course, we cannot see directly, but only by its shadow C. However, it is g and not C which obeys our criterion of simplicity, for g is a linear function of the new variable $f \in \mathscr{C}_0(A)$.



The shadow in the cave



The real thing in $\mathscr{C}_0^*(A)$
(a linear function of f).

§65. A HUMAN ANALOGY

We have remarked more than once on the close connections between the calculus of variations and a number of ideas that have had a profound influence on other parts of mathematics. The preceding section shows that these connections are not all limited to mathematics, and may take us into the much wider context of the great movements which have dominated human thought. We wish to pursue this a little further, and to gain in this way some insight into the nature and role of our dual space and of the curves that we have placed in it.

For this purpose, we shall speak of our curves as if they were not concepts, but beings like ourselves, or, better, as if they were human lives, which do indeed present a certain analogy to curves. We can perhaps imagine that in a variational problem it is perfection for a curve to attain the hoped for minimum, and that a curve twists and turns in its efforts to reach perfection. A reader may well object to this image on grounds that a curve is something that we draw or define by cold symbols, and that it is rather we ourselves who make it twist and turn with such an object. In this respect, we must request a mathematician's right to a little vagueness. Alternatively, we can use the objection itself to strengthen our analogy by appealing to a once popular limerick, which refers to man as "An engine that moves along predestinate grooves—not a bus, but a tram." According to this limerick our own striving is an illusion too. Still, there is no reason why we should not talk of striving, even if something else is really doing it for us, and if so, we should certainly allow ourselves the same freedom when speaking of curves. In any event, it is well within the conventions of language: how else can a cold drawing become a living portrait, or for that matter, an ordered collection of alphabet letters become words and thoughts? If these be fictions, then it is by such fictions that we live.

Thus, all in all, a human life is perhaps rather like a curve, striving as it does for an oft unattainable perfection: "The high that proved too high, the heroic for earth too hard, the passion that left the ground to lose itself in the sky."† The calculus of variations becomes then full of such striving; it is a very human calculus and our curves are very human curves.

† Robert Browning: *Dramatis Personae.* Abt Vogler, Stanza 10.